The Interference Rate of Radiation of Two Charges in Circular Motion

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Abstract

We present an exact formula for the computation of the interference rate of radiation in the case of two charges revolving with constant angular velocity at opposite ends of a diameter in a fixed circle. The formula is valid for arbitrary velocities of the charges, and can be easily studied by numerical methods, even for velocities very close to the velocity of light. For ultrarelativistic motion, the interference rate of radiation behaves as $\ln(1 - v^2/c^2)^{-1/2}$, which contrasts with the behavior $(1 - v^2/c^2)^{-2}$ for the rate of radiation for one charge in circular motion. This is the first exact calculation for the interference rate of radiation of two relativistic charges, and it is useful in connection with the old controversy about the correctness of the Lorentz-Dirac equations of motion for more than one charge.

I Introduction

In the case of a one charge in arbitrary motion, the total rate of radiation emitted at time t is given by the Larmor's formula, which reads

$$\frac{dW_{\rm rad}}{dt} = \frac{2}{3}\frac{e^2}{c}\gamma^4\left\{(\dot{\boldsymbol{\beta}})^2 + \gamma^2(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2\right\}, \tag{1.1}$$

where $\mathbf{v}$ is the charge velocity, c is the velocity of light, $\boldsymbol{\beta} = \mathbf{v}/c$, $\dot{\boldsymbol{\beta}} = d\boldsymbol{\beta}/dt$, and $\gamma = (1-\beta^2)^{-1/2}$. The simplicity of this formula, where the variables $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ are evaluated at the same time as the total rate of radiation, reflects some special properties of the point charge field. There are basically two derivations of the Larmor's formula (1.1). In one of them [1, 2], the total rate of radiation is first calculated in the Lorentz rest frame of the charge, and equation (1.1) is obtained using covariance arguments under Lorentz transformation. The other derivation is carried out directly in Minkowski space, using the fact that the radiation of a point charge can be characterized locally [3, 4, 5]. In particular, because of this property, it is not necessary to go very far from the charge in order to calculate the total rate or radiation.

In constrast with the one charge case, a formula for the total rate of radiation of two charges under arbitrary motion is not known yet. Furthermore, does not exist in the literature an exact formula for the total rate of radiation for any special type of motion of two charges. The source of the technical difficulty comes, of course, from the fact that the fields of the charges are explicit functions of the retarded times instead of the laboratory time. This point can also be illustrated by referring to the already mentioned derivations of (1.1). Thus, as emphazised by Landau & Lifschitz [2], for two charges there is generally no system of reference in which both charges are at rest simultaneously. And, on the other hand, the local characterization of radiation does not work for more than one charge[6].

Due to the superposition principle and the quadratic nature of the Poynting's vector, the energy flux for two charges contains three terms. Two of them are associated with the field of each individual charge separately. The third one corresponds to an interference term that mixes the fields of both charges. In the evaluation of the total rate of radiation, the first two terms give rise to a Larmor formula for each

charge; so the real problem is the calculation of the energy flux, across the surface of a sphere of very large radius, of the interference term.

There have been only a few attemps to calculate the total rate of radiation associated with the interference term. Huschilt and Baylis [7] studied this radiation in the case of two identical charged particles, that are moving in a straight line in head-on collision. This calculation is carried out with the help of the equation of motion for the charges, and in addition it involves some kind of non-relativistic approximations. Aguirregabiria and Bel [8] studied the radiation of two charges using a covariant formulation. These authors calculated an integral of the interference field over a circle, for a rather general motion of the charges. From this result, they elaborated a formalism, with the help of the equations of motion and some additional assumptions, in order to obtain succesive approximations for the total rate of radiation. The interference radiation has been studied also by Hojman et al [9], by means of a covariant formalism. However, these authors also introduce some kind of non-relativistic approximations.

In this paper we present an exact formula for the total rate of radiation in the case of two charges moving in a plane at the opposite ends of a diameter, revolving at constant angular velocity in a fixed circular orbit. The main motivation for this calculation is that it helps to solve an old controversy about the correctness of the Lorentz-Dirac equations of motion for more than one charge [10]. In fact, we have recently showed that, with appropriate external fields, the Lorentz-Dirac equations describes the circular motion under consideration in the case of two particles of equal charge and mass [11]. Then, if we know the total rate of radiation for this motion of the charges, we can check the consistency of the Lorentz-Dirac equations with the energy conservation law. We carried out such an analysis in [11], and showed that this type of circular motion allows us to see in a manifiest way the inconsistency of the Lorentz-Dirac equations for more than one charge with the energy conservation law.

The calculation is carried out directly in the laboratory frame. This is more appropiate than the covariant techniques, since the total rate of radiation can be clearly visualized from a physical point of view, and the result is free of any ambigui-

ties whatsoever. Our formula is an integral expression for the total rate of radiation. Unfortunately, the integral is too complicated for closed analytical evaluation. Nevertheless, it can be easily studied using numerical methods, even for velocities near the velocity of light. Furthermore, the integral can be evaluated approximately for ultrarelativistic motion of the charges. We find that the rate of interference radiation grows as $(\beta^4/4)\ \ln\gamma\pi$ when β tends to one. This result differs strongly with the behavior of the Larmor term of each charge, which behaves as $\beta^4\gamma^4$ when β is near one. Thus, for example, if we consider electrons of 500 Mev, the interference radiation is completely negligible in comparison with the Larmor term, since this latter is of the order of 10^{12}, while the interference term is near 1. For low velocities, however, it is known that the Larmor and the interference terms are comparable [12].

In section II we evaluate the energy flux across the surface of a sphere centered at the orbit center of the two charges, and show that this flux is independent of the time at which it is evaluated, as well as of the radius of the sphere. In section III we present a power series expansion in β of the total rate of radiation up to β^8. In section IV we derive an exact formula for the interference radiation term. In section V we present an analytical approximation of the exact formula for the case of ultra-relativistic motion.

II The interference rate of radiation

In the following we will be concerned only with the radiation of two charges moving in a plane at opposite ends of a diameter, revolving at constant angular velocity ω, in a fixed circular orbit of radius a. In this case, because of the symmetries of the motion, it is possible to identify without any ambiguities the total rate of radiation. With this purpose in mind we work directly in the laboratory frame, since this allows us to have a clear physical picture of the radiation.

FIGURE 1

As it is shown in figure 1, our coordinate system is such that its origin coincides with the center of the orbit, and the $X-Y$ plane is precisely the orbit plane. In this

figure we have drawn the positions of the charges at an arbitrary time t, and two spherical surfaces Σ_1, and Σ_2 centered at the origin of radii r_1 and r_2 respectively, with $r_2 > r_1 > a$, where a is the orbit radius. For a given time t, the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ change in a very complicated way from one point to another over the surface Σ_1; this is because the retarded times of the two charges, which are in general different, change in a complicated way with the position over Σ_1. In particular then, the Poynting vector $\mathbf{S} = (c/4\pi)\mathbf{E}\times\mathbf{B}$ is a complicated function over Σ_1, and so it is the total flux across Σ_1 at time t that we are interested in. Since the charges are moving jointly at constant angular velocity, the position of them at different times looks the same with respect to the whole surface Σ_1. This means that the energy flux across Σ_1, cannot depend on time. In Section IV we present a rigourous proof of this property, using the explicit form of the electromagnetic field of the two charges.

Now, if we denote by $u(\mathbf{x}, t)$ the energy density of the electromagnetic field, and by $\mathbf{S}$ the Poynting vector $(c/4\pi)\mathbf{E}\times\mathbf{B}$; then, since the domain Ω bounded by the two spherical surfaces Σ_1 and Σ_2 of Fig. 1 is free of charges, the following conservation law holds in it:

$$\nabla \cdot \mathbf{S} + \partial u/\partial t = 0. \tag{2.1}$$

From this equation we obtain

$$\int_{\Sigma_1} (\mathbf{S}\cdot\hat{\mathbf{r}})\, d\Sigma_1 - \int_{\Sigma_2} (\mathbf{S}\cdot\hat{\mathbf{r}})\, d\Sigma_2 = \frac{d}{dt}\int_{\Omega} u\,(\mathbf{x}, t)\, d^3x, \tag{2.2}$$

where $\hat{\mathbf{r}} = (\sin\theta\cos\varphi,\, \sin\theta\sin\varphi,\, \cos\theta)$ is the unit normal to Σ_1 and Σ_2. But, because of the symmetries of the motion of the two charges, it is clear that the total energy contained in Ω is independent of time. Therefore the integral

$$\int_{\Sigma} (\mathbf{S}\cdot\hat{\mathbf{r}})\, d\Sigma, \tag{2.3}$$

over the surface of the sphere of radius r is not only independent of time, but it is also independent of the radius r. In particular then, the integral (2.3) represents the total rate of radiation that escapes to infinity; and it can be evaluated over the surface of any sphere of arbitrary radius r, with $r > a$. Thus, for this special type of motion of the two charges,a reminiscence of the local characterization of the

radiation still survives, in the sense that it is not necessary to go to infinity in order to calculate the total rate of radiation.

We remark that the energy flux (2.3) is also independent of time and of the radius of Σ, in the case of only one charge in circular motion with constant velocity. Therefore, if we denote by $\mathbf{E}_1$, $\mathbf{B}_1$, and $\mathbf{E}_2$, $\mathbf{B}_2$ the electric and magnetic fields of the charges e_1 and e_2 respectively, the integral

$$(cr^2/4\pi)\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}(\mathbf{E}_1\times\boldsymbol{B}_2+\mathbf{E}_2\times\mathbf{B}_1)\cdot\hat{\mathbf{r}}\,d\varphi, \tag{2.4}$$

is independent of time and of the radius r. The integral (2.4) represents physically the rate of radiation due to the interference of the fields of both charges, and it will be discussed in detail in section IV.

III Power series representation for low velocities

The total rate of radiation of several charges can be in principle calculated by referring the retarded times of the charges, to the actual time t, by means of a power series expansion in $1/c$. The electric intensity of a group of charges can be represented by the following formula when r goes to infinity [12].

$$\begin{aligned}\mathbf{E} &= \frac{1}{rc^2}\left\{\left[\frac{d}{dt}\left(\sum_s e_s\mathbf{v}_s\right)+\frac{1}{1!c}\frac{d^2}{dt^2}\left(\sum_s\hat{\mathbf{r}}\cdot\mathbf{r}_s e_s\mathbf{v}_s\right)\right.\right.\\ &\quad\left.\left.+\cdots+\frac{1}{(n-1)!}\frac{1}{c^{n-1}}\frac{d^n}{dt^n}\left(\sum_s(\hat{\mathbf{r}}\cdot\mathbf{r}_s)^{n-1}e_s\mathbf{v}_s\right)+\cdots\right]\times\hat{\mathbf{r}}\right\}\times\hat{\mathbf{r}},\end{aligned} \tag{3.1}$$

where $\hat{\mathbf{r}}=\hat{\mathbf{i}}\sin\theta\cos\varphi+\hat{\mathbf{j}}\sin\theta\sin\varphi+\hat{\mathbf{k}}\cos\theta$, $\mathbf{r}_s$ and $\mathbf{v}_s$ denote the position and velocity of the charge e_s at time t. The far magnetic fields is $\mathbf{B}=\hat{\mathbf{r}}\times\mathbf{E}$.

In our case we have

$$\begin{aligned}\mathbf{r}_1(t) &= \hat{\mathbf{i}}a\cos\omega t+\hat{\mathbf{j}}a\sin\omega t,\\ \mathbf{r}_2(t) &= -\hat{\mathbf{i}}acos\omega t-\hat{\mathbf{j}}a\sin\omega t.\end{aligned} \tag{3.2}$$

For this motion we have computed the series (3.1) up to terms of power c^{-10}, and the result is given in appendix A. Using these formulae, the calculation of the

energy flux across the surface of a sphere of very large radius r is straightforward, but cumbersome. The result is the following

$$\begin{aligned}\frac{dW_{\rm rad}}{dt} &= \frac{2}{3}\frac{e_1^2 c}{a^2}\beta^4\left\{1+2\beta^2+3\beta^4+4\beta^6+5\beta^8+\cdots\right\}\\ &+\frac{2}{3}\frac{e_2^2 c}{a^2}\beta^4\left\{1+2\beta^2+3\beta^4+4\beta^6+5\beta^8+\cdots\right\}\\ &-\frac{4}{3}\frac{e_1e_2 c}{a^2}\beta^4\left\{1-\frac{14}{5}\beta^2+\frac{53}{7}\beta^4-\frac{18556}{945}\beta^6+\frac{515591}{10395}\beta^8+\cdots\right\}.\end{aligned} \tag{3.3}$$

The first two series correspond to power series expansions of the Larmor term of each charge. In fact, for a charge in circular motion with constant velocity, the Larmor formula (1.1) is reduced to

$$\frac{2}{3}\frac{e^2 c}{a^2}\beta^4(1-\beta^2)^{-2}. \tag{3.4}$$

The series

$$-\frac{4}{3}\frac{e_1e_2 c}{a^2}\beta^4\left\{1-\frac{14}{5}\beta^2+\frac{53}{7}\beta^4-\frac{18556}{945}\beta^6+\frac{515591}{10395}\beta^8+\cdots\right\}, \tag{3.5}$$

of (3.3) represents, of course, the rate of radiation associated with the interference between the fields of both charges. The first two terms of (3.5) are already known [12]. As we will show, the series (3.5) gives the total rate of interference radiation with an error less than 4% for $\beta < 0.2$. Clearly, the error increases with β, and a power series in β for the interference radiation is hopeless for β near 1, since a very large number of terms would be needed in this region.

IV An exact formula

We will consider now the energy flux of the interference term across the band between the angles θ and $\theta + d\theta$ over the surface of the sphere of radius r at time t, that is, the contribution

$$\int_0^{2\pi}(\mathbf{E}_1\times\mathbf{B}_2+\mathbf{E}_2\times\mathbf{B}_1)\cdot\hat{\mathbf{r}}\,d\varphi, \tag{4.1}$$

to the rate given in Eq. (2.4). The electric field $\mathbf{E}_1$ generated by the charge e_1 is given by the well-known Lienard-Wiechert formula [13]

$$\mathbf{E}_1(\mathbf{x},t) = e_1 \left[\frac{(\hat{\mathbf{n}}_1 - \boldsymbol{\beta}_1)(1-\beta_1^2)}{\kappa_1^3 R_1^2} \right]_{\text{ret}} + \frac{e_1}{c} \left[\frac{\hat{\mathbf{n}}_1}{\kappa_1^3 R_1} \times \left\{ (\hat{\mathbf{n}}_1 - \boldsymbol{\beta}_1) \times \dot{\boldsymbol{\beta}}_1 \right\} \right]_{\text{ret}} . \qquad (4.2)$$

The corresponding magnetic induction B_1 is

$$\mathbf{B}_1 = \hat{\mathbf{n}}_1 \times \mathbf{E}_1. \qquad (4.3)$$

In equation (4.2) and (4.3), $\hat{\mathbf{n}}_1$ is the unit vector that points from the retarded position $\mathbf{r}_1(t_1)$ of charge e_1, associated with the point $\mathbf{x}$ and time t, to the detection point $\mathbf{x}$. The vectors $\boldsymbol{\beta}_1$ and $\dot{\boldsymbol{\beta}}_1$ have already been defined in connection with formula (1.1); but now they have to be evaluated at the retarded position $\mathbf{r}_1(t_1)$ of the charge e_1. Moreover, R_1 represents the distance between the detection point $\mathbf{x}$ and the retarded position $\mathbf{r}_1(t_1)$ of charge e_1; and κ_1 denotes the following positive number

$$\kappa_1 = 1 - \hat{\mathbf{n}}_1 \cdot \boldsymbol{\beta}_1. \qquad (4.4)$$

The electric field $\mathbf{E}_2$ and magnetic induction $\mathbf{B}_2$ of the charge e_2 are given by (4.2) and (4.3) respectively, but where the quantities $\hat{\mathbf{n}}_2$, $\boldsymbol{\beta}_2$, $\dot{\boldsymbol{\beta}}_2$, R_2 and κ_2 are referred to the retarded time t_2 of the charge e_2 associated with the detection point $\mathbf{x}$ and time t.

In what follows we will carry out our calculations in the coordinate system shown in Fig. 2; where the positions of charges e_1 and e_2 are described by the vectors $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ defined in Eqs. (3.2), and the detection point is $\mathbf{x} = \hat{\mathbf{i}}\, r \sin\theta \cos\varphi + \hat{\mathbf{j}}\, r \sin\theta \sin\varphi + \hat{\mathbf{k}}\, r \cos\theta$. In Fig. 2 we have drawn the positions of the charges at three differents times, namely at t, t_1 and t_2; where t is the time at which we are going to calculate the flux across the surface of the sphere of radius $r > a$; the time t_1 corresponds to the retarded time of charge e_1 associated with $(\mathbf{x}, t)$, and t_2 is the retarded time of charge e_2 associated with $(\mathbf{x}, t)$.

FIGURE 2

Since the time interval $t - t_1$ that needs e_1 to go from its retarded position B_1 to the actual position A_1 is the same that takes the light for travel from B_1 to P,

we have

$$t - t_1 = (a/c)\xi^{-1}\left\{1 + \xi^2 - 2\xi\sin\theta\cos(\varphi - wt_1)\right\}^{1/2}. \qquad (4.5)$$

Similarly, since the time interval $t - t_2$ that needs e_2 to go from its retarded position C_2 to the actual position A_2, is the same that takes the light for travel from C_2 to P, we have

$$t - t_2 = (a/c)\xi^{-1}\left\{1 + \xi^2 + 2\xi\sin\theta\cos(\varphi - \omega t_2)\right\}^{1/2}, \qquad (4.6)$$

where ξ denotes the parameter

$$\xi = \frac{a}{r} < 1. \qquad (4.7)$$

Equations (4.5) and (4.6) are complicated functional equations that determine in an unique way the retarded times t_1 and t_2 respectively, as functions of the parameters t, r, θ and φ. The retarded times t_1 and t_2 are the same only for detection points over the z axis. Instead of working with the retarded times t_1 and t_2, it is convenient to introduce the following variables.

$$x = \varphi - \omega t_1, \qquad (4.8)$$

$$y = \varphi - \omega t_2. \qquad (4.9)$$

In the integral (4.1) the parameters t, r and θ are fixed; then Eq. (4.5) determines t_1 as a function of the angle φ, and therefore the variable x defined in Eq. (4.8) has a unique value for each φ in the interval $0 < \varphi < 2\pi$. This property allows us to carry out the integral (4.1) as an integral over the variable x. In order to see this clearly, let us first note that the correspondence between the variables x and y in Eqs. (4.8), and (4.9) is one to one. In fact, from Eqs. (4.5) and (4.6) we get the following relation between x and y

$$y - x = \beta\xi^{-1}\left\{\left(1 + \xi^2 + 2\xi\sin\theta\cos y\right)^{1/2} - \left(1 + \xi^2 - 2\xi\sin\theta\cos x\right)^{1/2}\right\} \qquad (4.10)$$

where $\beta = a\omega/c$. Taking the derivative with respect to x in Eq. (4.10), we obtain

$$\frac{dy}{dx} = \frac{1 + \rho_1^{-1}\beta\sin\theta\sin x}{1 + \rho_2^{-1}\beta\sin\theta\sin y}, \qquad (4.11)$$

where

$$\rho_1 = (1 + \xi^2 - 2\xi\sin\theta\cos x)^{1/2}, \qquad (4.12)$$

and

$$\rho_2 = (1 + \xi^2 + 2\xi \sin\theta \cos y)^{1/2}. \tag{4.13}$$

But

$$1 + \rho_1^{-1} \beta \sin\theta \sin x = \kappa_1 > 0, \tag{4.14}$$

and

$$1 + \rho_2^{-1} \beta \sin\theta \sin y = \kappa_2 > 0, \tag{4.15}$$

therefore

$$\frac{dy}{dx} = \frac{\kappa_1}{\kappa_2} > 0, \tag{4.16}$$

which proves that the correspondence between x and y is one to one. This property holds for any time t, radius r and angle θ.

Let us consider now the integral (4.1); where the time t, the radius r and the angle θ remain fixed. From Eq. (4.5) we obtain

$$-\omega \frac{dt_1}{d\varphi} = \left(\rho_1^{-1} \beta \sin\theta \sin x \right) \frac{dx}{d\varphi}; \tag{4.17}$$

if we combine this equation with Eq. (4.8), we get

$$\frac{dx}{d\varphi} = \frac{1}{\kappa_1} > 0, \tag{4.18}$$

where κ_1 is explicitly given in Eq. (4.14). Now, the equation $d\varphi/dx = \kappa_1 > 0$ tells us that φ is an strictly monotonous increasing function of x; so we can put the integral (4.1) in the following form

$$\int_{\alpha}^{2\pi+\alpha} (\mathbf{E}_1 \times \mathbf{B}_2 + \mathbf{E}_2 \times \mathbf{B}_1) \cdot \hat{\mathbf{r}} \kappa_1 dx, \tag{4.19}$$

where the parameter α is given by

$$\alpha = -\omega t_1(\varphi = 0) = -\omega t_1(\varphi = 2\pi), \tag{4.20}$$

which in general depends in a complicated way on the time t, the radius r and the angle θ of the band. When the integrand of (4.19) is explicitly evaluated by using the electric field (4.2) and the magnetic induction (4.3), with the corresponding expression for $\mathbf{E_2}$ and $\mathbf{B_2}$, it can be shown that the variables x and y appear only

as $\sin x$, $\cos x$, $\sin y$ and $\cos y$. The integrand is, of course, a function of the variable x only since, as shown above, y is uniquely determined by the value of x in Eq. (4.10). Moreover, as it can be easily seen, the correspondence between x and y is such that if y is the value associated with x, then $y+2\pi$ is the value associated with $x+2\pi$. Thus we conclude that the integrand of (4.19) is a periodic function of x, with a period of 2π. This property implies at once that the integral (4.19) does not depend on the value of the parameter α; so we can put $\alpha = 0$ in it, obtaining the following representation for the energy-flux (4.1).

$$\int_0^{2\pi} (\mathbf{E}_1 \times \mathbf{B}_2 + \mathbf{E}_2 \times \mathbf{B}_1) \cdot \hat{\mathbf{r}} \kappa_1 dx. \tag{4.21}$$

In particular then, the energy flux across the band between θ and $\theta + d\theta$ over the sphere of radius r does not depend on the value of the time at which it is evaluated. The integral (4.21) depends, however, in a very complicated way on the radius r and the angle θ.

The time-independence of the integral (4.21) is true for any band over the surface of the sphere of radius r; therefore the energy flux across the whole surface of the sphere is also independent of time. This property was inferred on symmetry grounds in section II. There we also proved that the interference of radiation given by

$$(c/4\pi) r^2 \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} (\mathbf{E}_1 \times \mathbf{B}_2 + \mathbf{E}_2 \times \mathbf{B}_1) \cdot \hat{\mathbf{r}} \kappa_1 dx \tag{4.22}$$

can be evaluated for an arbitrary radius r with $r > a$, because it is independent of r. The integrand of (4.22) contains a great number of terms for any finite value of r, and in addition Eq. (4.10) that links the variables x and y is very complicated for an arbitrary r. Strong simplifications of the integrand of (4.22) and of the functional relation (4.10) are obtained when considering the limit when r goes to infinity. We emphasize the fact that, due to the independence of the interference rate on the radius r, this limit does not present any complication, being perfectly well defined. In this limit Eq. (4.10) is reduced to

$$y - x = \beta \sin\theta (\cos y + \cos x), \tag{4.23}$$

and the interference rate of radiation (4.22) becomes

$$-\frac{4}{3}\frac{e_1 e_2 c}{a^2}\beta^4 I(\beta), \tag{4.24}$$

with $I(\beta)$ given by

$$I(\beta)=\frac{3}{4\pi}\int_0^{\pi/2}\sin\theta d\theta\int_0^{2\pi}\frac{(\cos^2\theta\cos x\cos y+\sin x\sin y-\beta^2\sin^2\theta)dx}{(1-\beta\sin\theta\sin x)^2(1+\beta\sin\theta\sin y)^3}. \tag{4.25}$$

If instead of changing the variable φ in the integral (4.1) by the x of Eq. (4.8), we perform the integral (4.1) by means of the variable y defined in Eq. (4.9), we would obtain in place of (4.25) the following expression for $I(\beta)$.

$$I(\beta)=\frac{3}{4\pi}\int_0^{\pi/2}\sin\theta d\theta\int_0^{2\pi}\frac{(\cos^2\theta\cos x\cos y+\sin x\sin y-\beta^2\sin^2\theta)dy}{(1-\beta\sin\theta\sin x)^3(1+\beta\sin\theta\sin y)^2}. \tag{4.26}$$

This formula looks different from (4.25), but it can be easily shown, with the help of Eq. (4.23), that both formulae are the same. We also point out that the functional Eq. (4.23) has the solutions

$$x=y=\pi/2 \tag{4.27}$$

and

$$x=y=3\pi/2 \tag{4.28}$$

independently of the value of β and θ.

Eq. (4.25), with $y(x)$ defined in Eq. (4.23), is an exact formula for the interference radiation, and it is the main result of this paper. Unfortunately, due to the complicated relation between the variables x and y defined by Eq. (4.23), the integral (4.25) cannot be evaluated in a closed analytical way. This situation contrasts with what happens when we consider the case of one charge in circular motion using the present treatment, where the corresponding integral can be explicitly done as it is shown in appendix B.

The integral (4.25) can be easily studied by means of numerical techniques. In Fig. 3 we show the function $y(x)$ in the orbit plane, for three different values of the parameter β; namely for $\beta=0$, $\beta=0.5$ and $\beta=0.9999$. In Fig. 4 we present the results of the numerical treatment of the function $I(\beta)$ of Eq. (4.25) in the interval

$0 \leq \beta < 0.99$ [14], where instead of the variable β we have used the variable ϕ defined by

$$\beta \cos\phi = \phi, \tag{4.29}$$

since it is more convenient. From the last equation we obtain

$$\frac{d\beta}{d\phi} = \frac{1+\beta\sin\phi}{\cos\phi}. \tag{4.30}$$

The condition $\beta < 1$ implies $\phi < 0.739$, so that $d\beta/d\phi > 0$. This shows that the correspondence between β and ϕ is one to one.

In figure 4 we have also drawn a dotted curve that represents the power series of Eq. (3.5)

FIGURE 3

FIGURE 4

V The interference rate in the ultrarelativistic case

When $\boldsymbol{\beta}$ is very close to one, the integrand of (4.25) is significative only for values of the variables x and y around

$$x_1 = y_1 = \pi/2 \tag{5.1}$$

$$x_2 = y_2 = 3\pi/2 \tag{5.2}$$

In the approximate evaluation of (4.25) we will use a procedure similar to that of reference [15] for the one charge case; but now the approximations are more crude because the integrand of (4.25) around (5.1) and (5.2) is not as sharply defined as in the one electron case. Nevertheless, this somewhat heuristic procedure allows us to obtain a simple analytical formula for the leading part of the interference rate, which accuracy improves according as $\boldsymbol{\beta}$ becomes close to one.

Since the radiation is mainly concentrated in the orbit plane, it is convenient to introduce the angle

$$\chi = \pi/2 - \theta \tag{5.3}$$

Then, we are going to consider an expansion of the integrand of (4.25) with $\delta x = x - x_1$ and χ of the order of γ^{-1}. From (4.23) it follows then that $\delta y = y - y_1$ is of the order of γ^{-3}, that is, y practically does not change when x is around $\pi/2$, and we can put $y = \pi/2$. In this way we can approximate the integral of x around x_1 in (4.25) by

$$\int\limits_{x_1-\delta x}^{x_1+\delta x} \frac{[(1/2)(x-x_1)^2 - \alpha^2]dx}{2[(x-x_1)^2+\alpha^2]^2} \tag{5.4}$$

where

$$\alpha^2 = \gamma^{-2}(1 + [\gamma\chi]^2). \tag{5.5}$$

Extending the limits of integration between $-\infty$ and $+\infty$ in (5.4), we obtain for it the value $\pi/8\alpha$. Now, since the radiation is mainly concentrated in the orbit plane, we can approximate $\sin\theta$ by 1 in the outermost integration of eq (4.25), and if the variable x in changed by $z = \gamma\chi$, we get the following contribution around $x_1 = \pi/2$, for the interference rate of radiation.

$$I_1(\beta) = \frac{3}{32} \int\limits_0^{\gamma\pi/2} \frac{dz}{(1+z^2)^{1/2}} = \frac{3}{32} \ln(\gamma\pi) \tag{5.6}$$

The contribution $I_2(\beta)$ around $x_2 = 3\pi/2$ of the integral (4.25) must be, on symmetry grounds, equal to (5.6). In this way, we get the following approximated formula for the interference rate of radiation in the ultrarelativistic case.

$$-\frac{4}{3}\frac{e_1 e_2}{a^2} c\,\tilde{I}(\gamma) \tag{5.7}$$

where

$$\tilde{I}(\gamma) = \frac{3}{16} \ln(\gamma\pi) \tag{5.8}$$

In table 5.1 we represent the interference rate given by the approximated formula (5.8), and the value of this quantity evaluated numerically from the exact formula (4.25). As expected, the accuracy of (5.8) improves when γ increases.

Table 5.1			
γ	$I(\gamma)$	$\tilde{I}(\gamma)$	Accuracy
100	1,235	1.0781	-12,7 %
500	1,536	1,3799	-10,16 %
1000	1,666	1,5098	-9,37 %
5000	1,968	1,8116	-7,95 %
10000	2,098	1,9416	-7,45 %
30000	2,304	2,1476	-6,79 %

The behavior (5.8) for the interference rate of radiation in the ultrarelativistic case contrasts strongly with the behavior of the Larmor terms of each charge, which behaves as γ^4 when γ goes to infinity. In particular, in the ultrarelativistic case, the interference rate of radiation is completely negligible in comparison with the Larmor terms of each charge.

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APPENDIX A

The far field produced by two charges in arbitrary motion can be expressed in the following form:

$$\mathbf{E} = \sum_{n=1}^{\infty} \mathbf{E}_n \tag{A.1}$$

where $\mathbf{E}_n$ is the last term in Eq. (3.1). Introducing the notation

$$\begin{aligned}
\mathbf{r} &= \mathbf{r}_1 = -\mathbf{r}_2 = a\cos\omega t\hat{i} + a\sin\omega t\hat{j} &\text{(A.2)}\\
\mathbf{v} &= \mathbf{v}_1 = -\mathbf{v}_2 = -a\omega\sin\omega t\hat{i} + a\omega\cos\omega t\hat{j} &\text{(A.3)}\\
\alpha &= \hat{\mathbf{r}}\cdot\mathbf{r} = ar\sin\theta\cos(\varphi-\omega t) &\text{(A.4)}\\
\eta &= \hat{\mathbf{r}}\cdot\mathbf{v} = a\omega r\sin\theta\sin(\varphi-\omega t) &\text{(A.5)}\\
\hat{\mathbf{r}} &= \sin\theta\cos\varphi\hat{i} + \sin\theta\sin\varphi\ hatj + \cos\theta\hat{k} &\text{(A.6)}
\end{aligned}$$

we obtain the following expressions for the fields:

$$\begin{aligned}
E_1 &= \frac{(e_1-e_2)\omega^2}{rc^2}\{\mathbf{r}-\alpha\hat{\mathbf{r}}\} &\text{(A.7)}\\
E_2 &= \frac{2(e_1+e_2)\omega^2}{rc^3}\{\eta\mathbf{r}+\alpha\mathbf{v}-2\alpha\eta\hat{\mathbf{r}}\} &\text{(A.8)}\\
E_3 &= \frac{(e_1-e_2)\omega^2}{2rc^4}\Big\{(6\eta^2-7\omega^2\alpha^2)\mathbf{r}+14\alpha\eta\mathbf{v}+(7\omega^2\alpha^3-20\alpha\eta^2)\hat{\mathbf{r}}\Big\} &\text{(A.9)}\\
E_4 &= \frac{4(e_1+e_2)\omega^2}{3rc^5}\Big\{(3\eta^3-12\omega^2\alpha^2\eta)\mathbf{r}+(12\eta^2\alpha-5\omega^2\alpha^3)\mathbf{v}\\
&\quad+(17\omega^2\alpha^3\eta-15\alpha\eta^3)\hat{\mathbf{r}}\Big\} &\text{(A.10)}\\
E_5 &= \frac{(e_1-e_2)\omega^2}{24rc^6}\Big\{(120\eta^4-1080\omega^2\eta^2\alpha^2+241\omega^4\alpha^4)\mathbf{r}+\\
&\quad(720\eta^3\alpha-964\omega^2\alpha^3\eta)\mathbf{v}+\\
&\quad(-840\eta^4\alpha+2044\omega^2\eta^2\alpha^3-241\omega^4\alpha^5)\hat{\mathbf{r}}\Big\} &\text{(A.11)}\\
E_6 &= \frac{(e_1+e_2)\omega^2}{120rc^7}\Big\{(720\eta^5-12000\omega^2\alpha^2\eta^3+8560\omega^4\alpha^4\eta)\mathbf{r}+\\
&\quad(6000\alpha\eta^4-17120\omega^2\alpha^3\eta^2+2256\omega^4\alpha^5)\mathbf{v}+\\
&\quad(-6720\alpha\eta^5+29120\omega^2\alpha^3\eta^3-10816\omega^4\alpha^5\eta)\hat{\mathbf{r}}\Big\} &\text{(A.12)}
\end{aligned}$$

$$
\begin{aligned}
E_7 &= \frac{(e_1-e_2)\omega^2}{720rc^8}\Big\{(5040\alpha^6 - 138600\omega^2\ alpha^2\eta^4 + 209790\omega^4\alpha^4\eta^2 - 19279\omega^6\alpha^6)\mathbf{r}+\\
&\quad (55440\alpha\eta^5 - 279720\omega^2\alpha^3\eta^3 + 115674\omega^4\alpha^5\eta)\mathbf{v} +\\
&\quad (-60480\alpha\eta^6 + 418320\omega^2\alpha^3\eta^4 - 325464\ omega^4\alpha^5\eta^2 + 19279\omega^6\alpha^7)\hat{\mathbf{r}}\Big\} \qquad (A.13)
\end{aligned}
$$

$$
\begin{aligned}
E_8 &= \frac{8(e_1+e_2)\omega^2}{315rc^9}\Big\{(-10150\omega^6\alpha^6\eta + 35280\omega^4\alpha^4\eta^3 - 13230\omega^2\alpha^2\eta^5 + 315\eta^7)\mathbf{r}+\\
&\quad (-1957\omega^6\alpha^7 + 30450\omega^4\alpha^5\eta^2 - 35280\omega^2\alpha^3\eta^4 + 4410\alpha\eta^6)\mathbf{v} +\\
&\quad (12107\omega^6\alpha^7\eta - 65730\omega^4\alpha^5\eta^3 + 48510\omega^2\alpha^3\eta^5 - 4725\alpha\eta^7)\hat{\mathbf{r}}\Big\} \qquad (A.14)\\
E_9 &= \frac{(e_1-e_2)\omega^2}{40320rc^{10}}\Big\{(2771521\omega^8\alpha^8 - 55597920\omega^6\alpha^6\eta^2 + 92786400\omega^4\alpha^4\eta^4 -\\
&\quad 22014720\omega^2\alpha^2\eta^6 + 362880\eta^8)\mathbf{r} +\\
&\quad (-22172168\omega^6\alpha^7\eta + 111195840\omega^4\alpha^5\eta^3 - 74229120\omega^2\alpha^3\eta^5 + 6289920\alpha\eta^7)\mathbf{v}\\
&\quad (-2771521\omega^8\alpha^9 + 77770088\omega^6\alpha^7\eta^2 - 203982240\omega^4\alpha^5\eta^4 + 96243840\omega^2\alpha^3\eta^6\\
&\quad -6652800\alpha\eta^8)\hat{\mathbf{r}}\Big\} \qquad (A.15)
\end{aligned}
$$

For circular motion the Pointying vector becomes

$$
\boldsymbol{S} = \frac{c}{4\pi}E^2\hat{\mathbf{r}} \qquad (A.15)
$$

and it can be evalued using the expressions of this appendix. If we now perform an integration over the surface of a sphere centered in the origin and that encloses the orbit, we obtain eq. (3.3).

APPENDIX B

In the case of one electron in circular orbit with constant velocity, the energy flux across the spherical surface or radius r and center at the orbit center, namely

$$(cr^2/4\pi)\int_0^\pi \sin\theta d\theta \int_0^{2\pi} (\mathbf{E}_1 \times \mathbf{B}_1)\cdot \hat{\mathbf{r}} d\varphi, \qquad (B.1)$$

is, like Eq. (2.4), independent of the time t and of the radius r of the sphere. On introducing in (B.1) the variable x of Eq. (4.8), and on taking the limit when r goes to infinity, it becomes

$$(ce^2/4\pi a^2)\beta^4 \int_0^\pi \sin\theta d\theta \int_0^{2\pi} \left\{ \frac{1}{(1-\beta\sin\theta\sin x)^3} - \frac{\gamma^{-2}\sin^2\theta\cos^2 x}{(1-\beta\sin\theta\sin x)^5} \right\} dx$$

These integrals, unlike the integrals of Eq. (4.25), can be easily evaluated in a closed analytical way. The result is, of course, Eq. (3.4).

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Figure and Table Captions

Table 5.1. Comparison between the numerical calculation of the exact expression (4.25), here denoted by $I(\gamma)$, and the asymptotic formula (5.8), denoted by $\tilde{I}(\gamma)$. We also show the porcentual error with respect to the numerical value. As expected, the approximation (5.8) improves as γ becomes larger.

Figure 1. Two charges e_1 and e_2 in circular motion at constant angular velocity ω. The orbit has radius a, and the coordinate axes have been chosen so that the orbit is centered at the origin and contained in the $X-Y$ plane. We also show two spherical surfaces Σ_1 and Σ_2 centered at the orbit's center, Ω being the domain bounded by Σ_1 and Σ_2.

Figure 2. At the observation, time t the charges e_1 and e_2 are ubicated at opposite ends of diameter A_1A_2. In the same way, at the retarded times t_1 and t_2 of charges e_1 and e_2 they are ubicated at the ends of the dotted diameter B_1B_2 and the dashed diameter C_1C_2 respectively. We also show the retarded distances R_1 and R_2 from the retarded positions of the charges to the observation point P, and the radius r of the spherical surface to which P belongs.

Figure 3. The curves represent the function $y(x)$ defined by the retardation condition, eq. (4.23). The plot is made for three different values of the parameter β, namely $\beta = 0$, $\beta = 0.5$ and $\beta = 0.9999$. Note that in all cases $x = y = \pi/2$ and $x = y = 3\pi/2$ are solutions of eq. (4.23).

Figure 4. The solid lines show the numerical evaluation of function $I(\beta)$ defined by eq. (4.25). The plot is made in terms of the variable ϕ defined by $\beta \cos\phi = \phi$, for the range $0 < \beta \leq 0.99$. We also show a dotted line that represents the power series expansion for the interference rate, eq. (3.5), in the range $0 < \beta \leq 0.4$.